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**Quantum model of interacting “strings” on the square lattice**H.E. Boos<sup>1</sup>Physikalisches Institut der Universität Bonn  
53115, Bonn, Germany**Abstract**

The model which is the generalization of the one-dimensional XY-spin chain for the case of the two-dimensional square lattice is considered. The subspace of the “string” states is studied. The solution to the eigenvalue problem is obtained for the single “string” in cases of the “string” with fixed ends and “string” of types (1,1) and (1,2) living on the torus. The latter case has the features of a self-interacting system and looks not to be integrable while the previous two cases are equivalent to the free-fermion model.

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# 1. Introduction

In the classical paper [1] Bethe proposed his ansatz for the eigenfunctions for the one-dimensional Heisenberg chain [2]. In the end of this paper he wrote that he intended to generalize his result for the high dimensional case. Unfortunately, it was not done. Of course, we do not pretend to solving this problem. In this paper we just intend to do some steps which are more or less in this direction. Namely, we would like to consider the model proposed by Stroganov [9] which can be thought as a simple generalization of the one-dimensional spin chain for a two-dimensional case. An idea is to consider some quantum mechanical model with locally interacting spins. A Hamiltonian is a sum of some tensor operator products over the neighbours on a square lattice instead of the one-dimensional ordinary spin-chain. Such a Hamiltonian can be considered as the lattice Hamiltonian for some statistical three-dimensional model. For instance, in paper [6] Baxter and Quispel obtained the Hamiltonian for the three-dimensional Zamolodchikov model [7, 8].

Of course, there are various ways to write down such a Hamiltonian. Here we consider only one of very simple ways to do it. The model is rather similar to the ordinary free-fermion model [4] and for some cases we really reproduce the known answers for the free-fermion model. But inspite of the simplicity of this model some of cases look not so trivial and even seem not to be integrable because of some self-interacting effects. Nevertheless, we are able to obtain some equation which we call "secular" equation because of it's similarity to the ordinary secular equation in quantum mechanics. In principle, the solution to this equation gives the spectrum of the model for case of the self-interacting "string" also.

The paper is organized as follows. In Section 2 we give a formulation of the model. In Subsection 2.1 we discuss a Hilbert space and introduce a Hamiltonian. In Subsection 2.2 we give a graphical interpretation which seems to be useful below. In Section 3 we adduce some simple examples. In Section 4 we consider a diagonalization problem of the Hamiltonian acting in some closed subspace of states which we call "string" states. In Subsection 4.1 we give a basic formulation of "string" states. In Subsection 4.2 we consider the diagonalization problem of the "string" Hamiltonian in a case of the "string" with fixed ends. In Subsection 4.3 we discuss the homology classes  $(m, n)$  of the single "string" living on the torus. Subsection 4.4 is devoted to

the solution of the spectrum problem of the "string" Hamiltonian in a case when the "string" belongs to the homology class (1,1). In Subsection 4.5 we consider a case of the "string" from the homology class (1,2). In Section 5 we discuss the results and possible ways of a further progress. In Appendix we give the detailed derivation of the result for the "string" of type (1,2) in a case of the  $n \times 2$  lattice with cyclic boundary conditions.

## 2. Formulation of the model

### 2.1. The quantum state space and the Hamiltonian

As it was mentioned in Introduction the model [9] we would like to consider is some quantum mechanical model of two dimensional system of locally interacting spins.

Let  $\mathcal{L}$  is simple quadratic  $m \times n$  lattice with the toroidal topology. Let us enumerate each site of this lattice by the pair of integers  $(i, j)$  where  $0 \leq i \leq m - 1, 0 \leq j \leq n - 1$ . This enumeration is shown in Fig.1 for some part of the lattice  $\mathcal{L}$ .

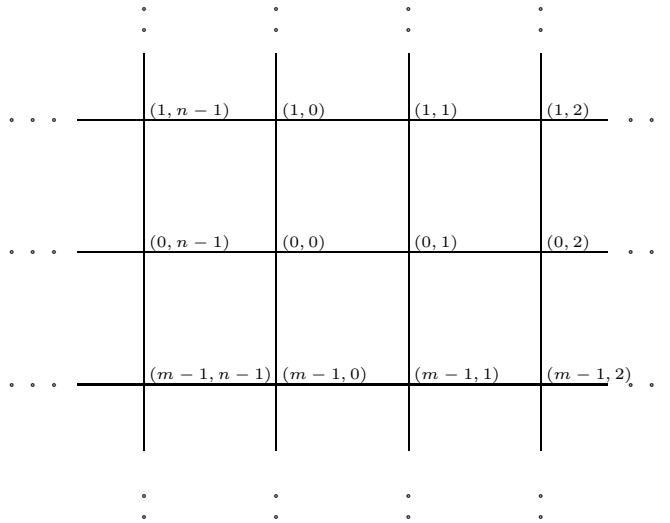


Fig. 1. Enumeration of the sites.

Let us assign some "spin" variables with two possible values (for example,

0 and 1) to each of  $2mn$  edges of the lattice  $\mathcal{L}$ . Let us denote spin variables  $\sigma_{i,j}$  for horizontal edges and  $\eta_{i,j}$  for vertical ones as it is shown in Fig. 2.

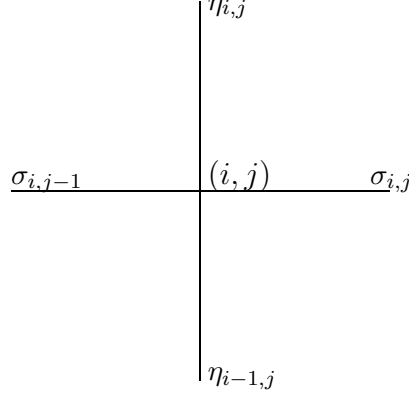


Fig. 2. Enumeration of the spins.

The state space  $A$  of our system can be considered as the direct product of  $2mn$  two dimensional complex spaces  $G_{i,j}$  and  $V_{i,j}$ :

$$A = \prod_{i=0}^{m-1} \prod_{j=0}^{n-1} (\otimes G_{i,j} \otimes V_{i,j}).$$

Let us choose  $2^{2mn}$  of all possible direct products of the basis vectors for the spaces  $G_{i,j}$  and  $V_{i,j}$  as the basis vectors for the space  $A$ .

One can choose the basis vectors in such a way that Pauli matrices have their standard form:

$$\sigma_{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\mathbf{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{\mathbf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{\mathbf{0}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The model under consideration is local in a sense that the corresponding Hamiltonian is decomposed into the sum of the terms which depend on the fixed number of those spin variables which belong to the nearest edges.

Namely, we have

$$H = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} h_{i,j}, \quad (2.1)$$

where

$$h_{i,j} = h(\sigma_{i,j}; \eta_{i,j}; \sigma_{i+1,j}; \eta_{i,j+1}) \quad (2.2)$$

The spins taking part in the elementary interaction correspond to the edges of the elementary sell (square) of the lattice  $\mathcal{L}$  (see Fig. 3).

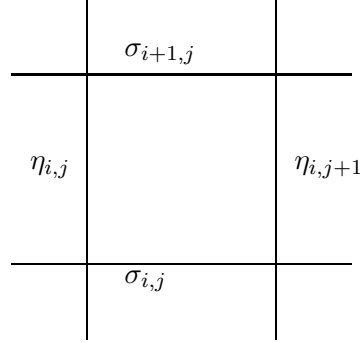


Fig. 3. Interacting spins.

Now we are ready to fix the exact form of the elementary interaction:

$$h(\sigma_{(1)}, \sigma_{(2)}, \sigma_{(3)}, \sigma_{(4)}) = e * (\sigma_{(1)}^- \sigma_{(2)}^+ \sigma_{(3)}^+ \sigma_{(4)}^- + \sigma_{(1)}^+ \sigma_{(2)}^- \sigma_{(3)}^- \sigma_{(4)}^+), \quad (2.3)$$

where  $\sigma^\pm = \sigma^x \pm i\sigma^y$  and the subscripts in the low brackets correspond to the four two dimensional spaces:  $G_{i,j}$ ,  $V_{i,j}$ ,  $G_{i+1,j}$ ,  $V_{i,j+1}$ .

## 2.2. Graphic interpretation

Let us remind the reader that for the basis of state space  $A$  one can choose  $2^{2mn}$  of the eigenvectors for all spin variables  $\sigma_{i,j}$  and  $\eta_{i,j}$ . Graphically one can represent each set of these vectors on the lattice  $\mathcal{L}$  by the colouring all the edges in one of two possible colours. The colour of each edge is determined by the eigenvalue of the spin variable which corresponds to this edge. Some example of such a colouring for a part of the lattice is shown in Fig. 4.

Instead of two possible colours we shall speak about the presence or the absence of colour on the corresponding edge of the lattice.

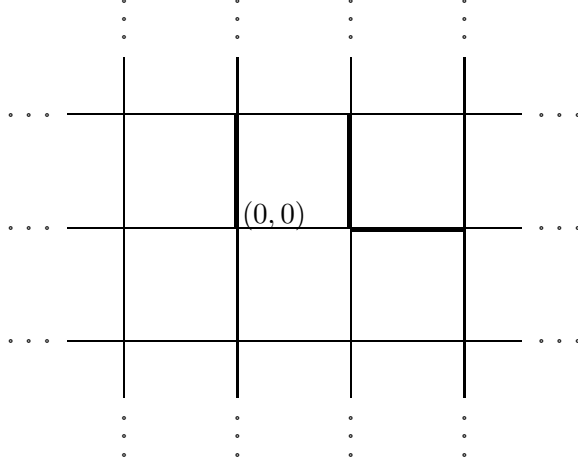


Fig. 4.

The basis vector  $|v\rangle$  which corresponds to Fig. 4 obeys to the following conditions:

$$\begin{aligned} \sigma_{00} |v\rangle &= 0, & \sigma_{01} |v\rangle &= |v\rangle, & \eta_{00} |v\rangle &= |v\rangle, \\ \eta_{01} |v\rangle &= |v\rangle, & \sigma_{10} |v\rangle &= 0, & \sigma_{11} |v\rangle &= 0, \end{aligned}$$

Let us consider the question of how the Hamiltonian  $H$  defined by formulae (2.1-2.3) acts on the basis vectors. Elementary interaction  $h_{i,j}$  depends on those spins which correspond to the edges of the elementary cell (square) of the lattice in accordance with the formula 2.3 (see Fig. 3).

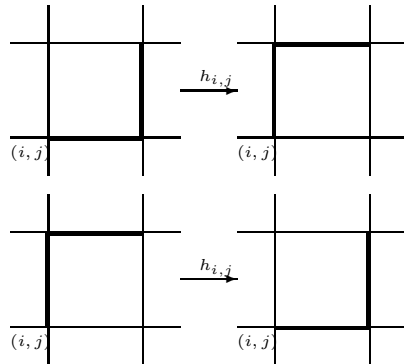


Fig. 5. The action of the Hamiltonian  $h_{i,j}$ .

The operators  $\sigma^+$  “give the colour” to the corresponding edge without

colour and annihilate the state which corresponds to the coloured edge. In contrast, the operators  $\sigma^-$  “remove the colour” from the coloured edge and annihilate the state which corresponds to the edge without colour. The action of the  $h_{i,j}$  on the colouration of the elementary square (sell) can be illustrated by Fig. 5.

### 3. Some simple examples

1. There are a lot of quantum states corresponding to some special colouring of the lattice which are “unmovable”, i.e. the Hamiltonian acts on such a states trivially just annihilating them. The simplest example of such a state is a straight line or the set of straight lines not intersecting with each other or intersecting in the end points as it is shown in Fig. 6

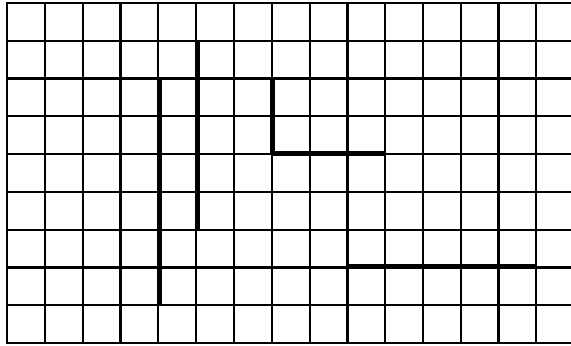


Fig. 6

Some other examples of “unmovable” states are shown in Fig. 7

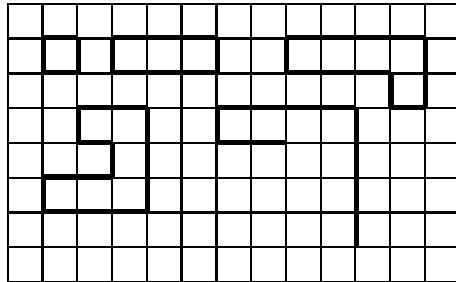


Fig. 7

2. Now let us adduce some simple examples of quantum states on which the Hamiltonian acts non-trivially. In fact, the simplest case of such a states is shown in Fig 5. Namely, in this case there are only two possible quantum states, say  $|0\rangle$  and  $|1\rangle$ , which correspond respectively to the right and left sides of the first line in Fig. 5 where only two edges of the whole lattice are coloured. The parameter  $e$  which is the single dimensional parameter in our problem (see (2.3)) can be equated to unity without loss of generality. Then the Hamiltonian  $H$  just interchanges these two states:

$$\begin{aligned} H |0\rangle &= |1\rangle \\ H |1\rangle &= |0\rangle \end{aligned} \quad (3.1)$$

It is very easy to find the eigenvalues  $E$  of the Hamiltonian in this case

$$E = \pm 1. \quad (3.2)$$

As the next case let us consider “one-step wave” which is shown in Fig. 8

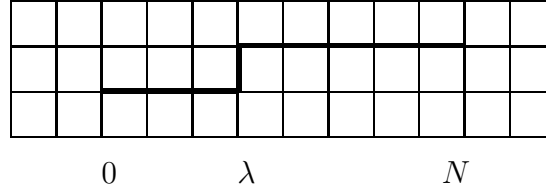


Fig.8 "One-step wave"

Let  $N$  be the “distance” between the end points of this “wave” and  $\lambda$  is the coordinate of the “jump” (in Fig. 8  $\lambda = 3, N = 8$ ). Let us denote this state  $|\lambda\rangle$ . It is easy to see that the Hamiltonian acts on this state as follows:

$$H |\lambda\rangle = |\lambda+1\rangle + |\lambda-1\rangle \quad (3.3)$$

To solve the eigenvalue problem  $H |\Psi\rangle = E |\Psi\rangle$  we can try to do the following substitution



$$| \Psi \rangle = \sum_{\lambda=0}^N \Psi(\lambda) | \lambda \rangle . \quad (3.4)$$

The static Schrödinger equation for the wave function looks very simple

$$E\Psi(\lambda) = \Psi(\lambda + 1) + \Psi(\lambda - 1). \quad (3.5)$$

Two states  $\lambda = 0$  and  $\lambda = N$  are the “first” and “last” states of the wave. Hence, we have the following ”boundary” conditions for the wave function

$$\Psi(-1) = \Psi(N + 1) = 0. \quad (3.6)$$

The eigenvalues  $E$  and eigenfunctions of the Hamiltonian can be easily found

$$E = x + x^{-1} \quad (3.7)$$

and

$$\Psi(\lambda) = x^{\lambda+1} - x^{-\lambda-1}. \quad (3.8)$$

In order to satisfy the conditions (3.6) we should set  $x$  to be some root of unity

$$x^{2N+4} = 1 \quad (3.9)$$

or

$$x = e^{\frac{i\pi k}{N+2}} \quad (3.10)$$

and  $k = 1, \dots, N + 1$ . Then  $E = 2 \cos \frac{i\pi k}{N+2}$ . The previous case corresponds to  $N = 1$ . Hence we have only two possibilities  $k = 1$  and  $k = 2$ . In the first case  $E = 2 \cos \frac{i\pi}{3} = 1$ . In the second case  $E = 2 \cos \frac{2i\pi}{3} = -1$  in accordance with (3.2).

We do not intend to classify here the whole space of quantum states. Below we consider only some important subspace of states which we call ”string” states.

## 4. The string spectrum

### 4.1. “Ice condition” and “string” states

Let us consider the subspace of the state space which is the linear span of some special subset of the basis vectors. Namely, let us consider all basis

vectors for which the corresponding colouring of the lattice  $\mathcal{L}$  satisfies to the so-called “ice condition” (see, for example, [3]). In other words, there are only six types of the allowed colouring of the edges which are adjacent with one site of the lattice (six types of the vertices in accordance with the standard terminology of statistical models on the squared lattice), see Fig. 9.

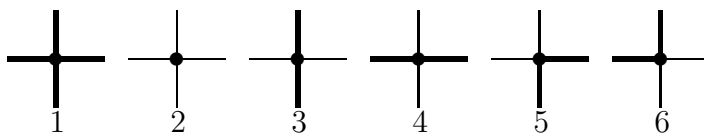


Fig. 9. Ice condition for the verticies.

The dimension of this subspace  $\mathcal{A}$  is equal to the number of the ways to colour the lattice  $\mathcal{L}$  taking into account the ice condition. This quantity is connected with the entropy of the “ice” model which was calculated in the thermodynamic limit by Lieb [5].

The action of the Hamiltonian  $H$  conserves the ice condition. In other words, the subspace  $\mathcal{A}$  is invariant under the action of  $H$ . Our main problem is to find the eigenvalues of the Hamiltonian  $H$  on the subspace  $\mathcal{A}$ .

The subspace  $\mathcal{A}$  in it's turn is the direct sum of the invariant subspaces  $\mathcal{A}^{(k)}$  where  $k = 0, 1, 2, \dots$  are nonnegative integers.

Any configuration which satisfies the ice condition can be represented as a set of continuous nonintersecting ways which go only up and to the left (see, for example, [3]).

Let us call this ways “stings”. Also, let us denote  $\mathcal{A}^{(k)}$  the subspaces which are spanned on the vectors corresponding to the colouring the lattice by  $k$  ways (or stings). We call  $\mathcal{A}^{(k)}$  the state space with  $k$  strings.

It is easy to see that all subspaces  $\mathcal{A}^{(k)}$  are invariant under the action of the Hamiltonian  $H$ . So, the number of strings  $k$  is the conservation number.

Since the subspace  $\mathcal{A}^{(0)}$  consists of only one vector  $|0\rangle$  which is annihilated by the Hamiltonian  $H$   $|0\rangle = 0$  let us begin the solving our problem of the diagonalization of  $H$  from the subspace  $\mathcal{A}^{(1)}$ . So, let us consider the

spectrum for the single string.

## 4.2. The string with the “fixed” ends

In Section 3 we considered the case of the “one-step wave”. In fact, this is a simple example of a string with the fixed ends when the string has only one “jump”. Suppose that the size of the lattice is big enough to consider more general case when the string has  $N$  “jumps”. The quantum state can be defined by fixing the coordinates  $\lambda_1, \lambda_2, \dots, \lambda_M$  of these jumps. Due to the “ice” condition these coordinates should satisfy the ordering:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_M \leq N \quad (4.1)$$

which corresponds to some partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_M)$ .<sup>2</sup> Some example of the string state for the case  $N = 9$  is shown in Fig. 10

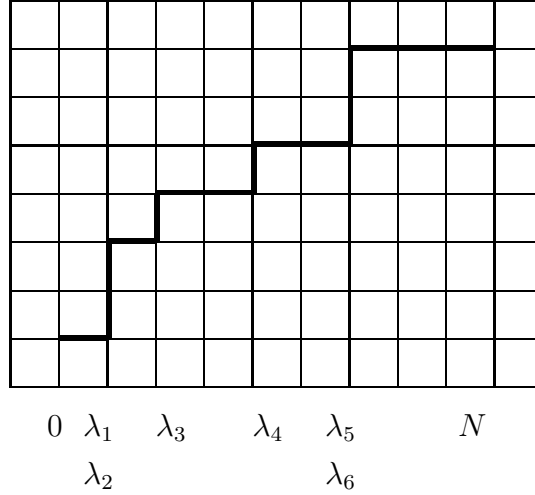


Fig.10 The string corresponding to the partition  $(1, 1, 2, 4, 6, 6)$  in the case  $N = 9$

Let us fix  $N$  and  $M$ . It is not difficult to calculate the dimension of the quantum space of the single string with the fixed ends. To do it let us

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<sup>2</sup>see for example the book [10]

consider  $\bar{\lambda}_j = \lambda_j + j$  which satisfy the chain of the strict inequalities instead of the non-strict ones given by (4.1)

$$1 < \bar{\lambda}_1 < \bar{\lambda}_2 < \dots < \bar{\lambda}_M < N + M + 1. \quad (4.2)$$

So, the problem is reduced to the calculating the number of the different ways to put  $M$  undistinguishable objects on the  $N + M$  places. The answer is well known. It is the binomial coefficient  $\binom{N+M}{M}$ . Therefore we have for a dimension of the quantum space

$$\dim = \binom{N+M}{M}. \quad (4.3)$$

It is easy to see that the Hamiltonian acts on some arbitrary string state  $\lambda$  as follows

$$H | \lambda > = \sum_{i=1}^M | \lambda + \delta_i > + | \lambda - \delta_i > \quad (4.4)$$

where  $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$  is  $M$ -component vector which has 1 on the  $i$ -th place and zeros on other places.

Now let us consider the wave functions instead of the basis vectors. As in Section 3 we are going to find the eigenvectors for the Hamiltonian  $H | \Psi > = E | \Psi >$  in the following form:

$$| \Psi > = \sum_{\lambda_1, \dots, \lambda_M} \Psi(\lambda) | \lambda > \quad (4.5)$$

where  $\Psi(\lambda)$  is short notation for the wave function  $\Psi(\lambda_1, \dots, \lambda_N)$ . The Schrödinger equation in the general position looks as follows:

$$E \Psi(\lambda) = \sum_{i=1}^M \Psi(\lambda + \delta_i) + \Psi(\lambda - \delta_i). \quad (4.6)$$

Of course, the states in the RHS of (4.4) or (4.6) should also correspond to some partition with the ordering (4.1). We can formally satisfy this requirement by equating to zero the wave functions which correspond to non-existing states  $| \lambda >$  with  $\lambda_{i+1} = \lambda_i - 1$ . As in the case of the “one-step wave” the ends of the string can not “move”. We can again formally equate the wave functions with  $\lambda_1 = -1$  and  $\lambda_M = N + 1$  to zero. So, we have

$$\Psi(\lambda) = 0, \quad \text{if} \quad \left\{ \begin{array}{l} \lambda_{i+1} = \lambda_i - 1 \text{ for at least one position } i \\ \lambda_1 = -1 \\ \lambda_M = N + 1 \end{array} \right. \quad (4.7)$$

It is easy to see that for the string of the general position  $0 < \lambda_1 < \dots < \lambda_M$  the monomial solution

$$\Psi(\lambda) = x^\lambda = x_1^{\lambda_1} \dots x_M^{\lambda_M} \quad (4.8)$$

satisfies the eigenvalue problem for the Hamiltonian (4.4) with the eigenvalue (or energy)

$$E = \sum_{i=1}^M x_i + x_i^{-1} \quad (4.9)$$

which is invariant under any permutation of the variables  $x_i$  and a substitution  $x_i \rightarrow x_i^{-1}$ . Of course, the monomial solution (4.8) does not satisfy the conditions (4.7). To find the right solution to the eigenvalue problem we have to consider a Bethe ansatz-like linear combination

$$\sum_{\sigma, \epsilon_1, \dots, \epsilon_M} A_\sigma(\epsilon_1, \dots, \epsilon_M) x_{\sigma(1)}^{\epsilon_1 \lambda_1} \dots x_{\sigma(M)}^{\epsilon_M \lambda_M} \quad (4.10)$$

where the sum is over all permutation  $\sigma$  and all possible signs  $\epsilon_i = \pm 1$ . One can check that the following simple determinant solution

$$\Psi(\lambda) = \text{Det}([x_i^{\lambda_j + j}])_{1 \leq i, j \leq M} \quad (4.11)$$

where  $[x] = x - x^{-1}$  belongs to the class of polynomials given by (4.10) and satisfy almost all conditions (4.7) with the exception of the last one in (4.7). Indeed, when we try to disturb the order of the sequence of parameters  $\lambda_i$  we get the pair of the same columns. So, the unwanted moves are suppressed automatically!

Hence, we have to satisfy the conditions  $\Psi(\lambda) = 0$  for  $\lambda_M = N + 1$ . So, we come to the quantization condition

$$x_i^{2(N+M+1)} = 1 \quad (4.12)$$

which generalizes the condition (3.9) for the "one-step wave" case. To enumerate all eigenvectors we should set

$$x_i = \Omega^{k_i} \quad (4.13)$$

where  $\Omega = e^{i\pi/(N+M+1)}$  and the wave numbers  $k_i$  satisfy the inequalities

$$0 < k_1 < k_2 < \dots < k_M < N + M + 1. \quad (4.14)$$

This chain of the inequalities coincides with the chain for  $\bar{\lambda}_j$  given by (4.2). Hence, the number of possible values  $k_i$  i.e. the number of all eigenvectors coincides with the dimension of the quantum space given by formula (4.3) as it should be.

The corresponding values of the energy  $E$  are still given by (4.9) which can be written in the following form

$$E = \sum_{j=1}^M 2 \cos \frac{\pi k_j}{N + M + 1}. \quad (4.15)$$

The appearance of the determinant in formula (4.11) for the eigenfunctions tells us about the connection of this problem with the free-fermion model or  $XY$ -chain [4].

In the next section we consider the spectral problem for the single string on the lattice with toroidal boundary conditions.

### 4.3. The ring of homologies of the single string on a torus

Let us consider the arbitrary single string configuration on the  $m \times n$  lattice torus  $\mathcal{L}$ . Let us apply to the figure 11

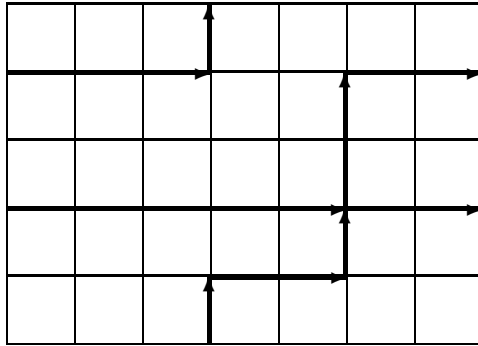


Fig. 11

One can think that we see the pieces of the three strings if to consider this figure as the part of some bigger lattice. But it turns out to be obvious that it is one string living on the  $5 \times 7$  torus. This string winds round the torus  $\overline{m} = 2$  times on the horizontal direction and  $\overline{n} = 1$  times on the vertical direction.

The Hamiltonian being local does not change the winding number of the string. So, the space  $\mathcal{A}^{(1)}$  decomposes into the direct sum of the invariant subspaces  $a_{\overline{m},\overline{n}}$ .

What can the integers  $\overline{m}$  and  $\overline{n}$  be ? Let  $\overline{m} = 0$  then  $\overline{n} = 1$  and we have  $n$  of the trivial string configurations of the following form:

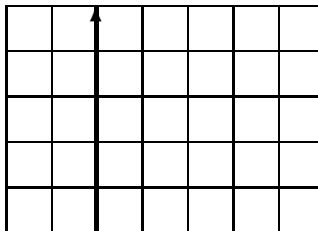


Fig. 12. Simple winding of the type (0,1)

In the same way we have  $\overline{m} = 1$  if  $\overline{n} = 0$  and we have  $m$  of the trivial horizontal configurations.

Let us note without the proof the fact that the space  $\mathcal{A}^{(1)}$  decomposes into the direct sum of  $mn + 2$  invariant subspaces  $a_{0,1}, a_{1,0} \dots a_{\overline{m},\overline{n}}$  where  $\overline{m}$  and  $\overline{n}$  satisfy the following inequalities:

$$1 \leq \overline{m} \leq m, \quad 1 \leq \overline{n} \leq n.$$

In the next section we shall consider the simplest nontrivial case of the string of the type (1,1). This string winds the torus one time in the horizontal direction and one time in the vertical one.

We shall see that the string in this case is also equivalent to the XY-chain. It will be also equivalent to the model considered by Bethe if to add the diagonal part to the Hamiltonian  $H$ .

#### 4.4. The spectrum of the string of type (1,1)

So, let us consider the subspace  $a_{1,1}$ . One can split all configurations of the string on  $m$  of classes which depend on the number of the horizontal line in the torus where the string leaves or intersects the vertical line with the number 0. The number of the class will be denoted as  $\alpha$ :

$$\alpha = 0, 1, \dots, m - 1.$$

Fig. 12 corresponds to the case  $\alpha = 1$  (for the torus  $4 \times 7$ ).

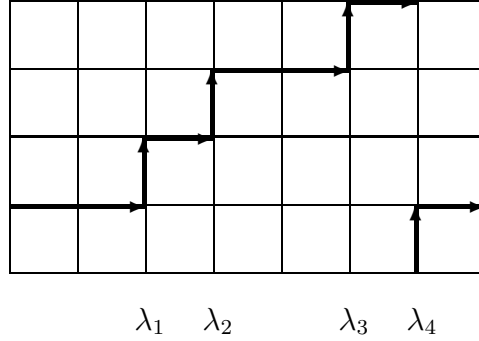


Fig. 13. The marking of the configuration (1,1)

Again the configuration for some fixed  $\alpha$  is characterized by  $m$  of the integers:

$$1 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq n \quad (4.16)$$

where as in the case of the string with fixed end points  $\lambda_i$  is the number of those vertical line where the string begins it's "climbing up" on the  $i$ -<sup>th</sup> sell from the horizontal line with the number  $\alpha$ . For example, Fig. 12 corresponds to the following set  $\lambda_1 = 2$ ,  $\lambda_2 = 3$ ,  $\lambda_3 = 5$  and  $\lambda_4 = 6$  i.e. to the partition (2, 3, 5, 6).

To calculate the whole number of the single string configurations we can apply a similar trick as in the case of string with the fixed ends. Namely, the chain of the non-strict inequalities (4.16) is equivalent to the following chain of the strict inequalities:



$$0 < \lambda_1 < \lambda_2 + 1 < \lambda_3 + 2 < \dots < \lambda_m + m - 1 < n + m \quad (4.17)$$

Hence, we can apply the same combinatorial calculation as it was done for derivation of formula (4.3). Taking into consideration that the variable  $\alpha$  takes  $m$  different values we get the dimension of the quantum space of the string of type (1,1):

$$\dim(a_{1,1}) = \frac{(m+n-1)!}{(n-1)!(m-1)!}. \quad (4.18)$$

As it was mentioned above the calculation of all possible configurations with any number of the strings is equivalent to the problem of the calculation of the entropy in the ice model on the square lattice.

Now we return to the problem of the diagonalization of the Hamiltonian  $H$ . Let us introduce the following notations for the basis vectors:

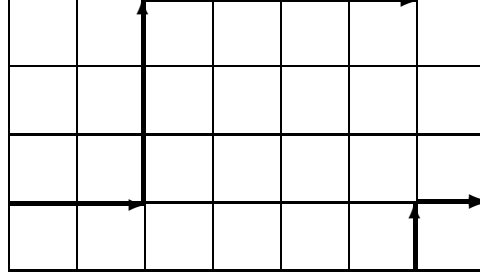
$$|\alpha; \lambda\rangle = |\alpha; \lambda_1, \lambda_2, \dots, \lambda_m\rangle. \quad (4.19)$$

This vector corresponds to the single string configuration of the type (1,1) from the class  $\alpha$  which is characterized by the parameters  $\lambda_1, \lambda_2, \dots, \lambda_m$ .

Let us say that the parameters  $\lambda_i$  are on the general position if none of equalities in the chain of inequalities (4.16) is fulfilled. The action of the Hamiltonian on the states parameterized by the parameters  $\lambda_i$  in the general position is given by a similar to (4.4) formula

$$H |\alpha; \lambda\rangle = \sum_{i=1}^m |\alpha; \lambda + \delta_i\rangle + |\alpha; \lambda - \delta_i\rangle. \quad (4.20)$$

Then suppose there is some group of the coinciding parameters  $\lambda_j, \lambda_{j+1}, \dots, \lambda_k$  where  $j < k$  but  $\lambda_j > 1$  and  $\lambda_k < n$ . This configuration corresponds to the presence of the jump of the length  $k - j + 1$  (see Fig. 14).



$\lambda_1 = \lambda_2 = \lambda_3$   $\lambda_4$   
 Fig. 14. The case of coinciding  $\lambda_i$

For this group of the parameters evidently only the decrease of  $\lambda_j$  and the increase of  $\lambda_k$  on the unity are allowed. Let us note that after this the parameter  $\alpha$  is not changed.

Now let us consider the situation which is connected with the fact that our choice of the state marking is not translational invariant. Namely, let  $\lambda_1 = 1$ . Then there is such a move of the string which increases  $\alpha$  on the unity:

$$| \alpha; 1, \lambda_2, \dots, \lambda_m > \Rightarrow | \alpha + 1; \lambda_2, \lambda_3, \dots, \lambda_m, n > . \quad (4.21)$$

We have the analogous situation on the right “edge” of the torus when the last parameter  $\lambda_m = n$ . In this case we have:

$$| \alpha; \lambda_1, \dots, \lambda_{m-1}, n > \Rightarrow | \alpha - 1; 1, \lambda_1, \dots, \lambda_{m-1} > . \quad (4.22)$$

Now let us consider the operator  $U$  which makes the shift up of the all configuration on the one sell of our lattice. This operator has the following properties:

- 1) It acts only on the index  $\alpha$ :

$$U | \alpha; \lambda > = | \alpha + 1; \lambda > , \quad (4.23)$$

- 2) It commutes with the Hamiltonian  $H$ ,
- 3)  $U^m = \mathbf{1}$

Due to the second property it is useful to consider another basis which can be obtained from the former one by the discrete Fourier transformation

in respect to the parameter  $\alpha$ :

$$| \lambda_1, \dots, \lambda_m; a > \equiv \sum_{\alpha=0}^{m-1} e^{2\pi i a \alpha / m} | \alpha; \lambda_1, \dots, \lambda_m >, \quad (4.24)$$

where

$$a = 0, \dots, m-1. \quad (4.25)$$

Now instead of the equations (4.21 and 4.22) we have:

$$| 1, \lambda_2, \dots, \lambda_m; a > \Rightarrow e^{-2\pi i a / m} | \lambda_2, \dots, \lambda_m, n; a > \quad (4.26)$$

and the conjugated equation:

$$| \lambda_1, \dots, \lambda_{m-1}, n; a > \Rightarrow e^{2\pi i a / m} | 1, \lambda_1, \dots, \lambda_{m-1}; a >. \quad (4.27)$$

Below we shall omit the parameter  $a$  from the marking of our new basis vectors.

Now let us consider the wave functions instead of the basis vectors. As it was done above we introduce the following expansion

$$| \Psi > = \sum_{\lambda} \Psi(\lambda) | \lambda > \quad (4.28)$$

and consider the Schrödinger equation

$$E \Psi(\lambda) = \sum_{i=1}^m \Psi(\lambda + \delta_i) + \Psi(\lambda - \delta_i). \quad (4.29)$$

Then we can repeat the same arguments which were applied for derivation of the determinant formula (4.11). As appeared we can use even more simple substitution for the wave function

$$\Psi(\lambda) = \sum_{\sigma} A_{\sigma} x_{\sigma(1)}^{\lambda_1} \dots x_{\sigma(m)}^{\lambda_m}.$$

In some sense it is the simplest variant of the Bethe ansatz.

As it was discussed above for the case when  $H$  acts on the state with some group of the coinciding parameters  $\lambda_j = \lambda_{j+1} = \dots = \lambda_k$ ,  $j < k$  we have only two allowed moves which change this group  $\lambda$ . Namely, the parameter

$\lambda_j$  can be decreased on unity while  $\lambda_k$  can be increased on the unity. As above this statement allows us to fix the coefficients  $A_\sigma$  in the last expression for the wave function. The result is rather simple:

$$\Psi(\lambda_1, \lambda_2, \dots, \lambda_m) = \text{const} \times \text{Det} \begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2+1} & \dots & x_1^{\lambda_m+m-1} \\ x_2^{\lambda_1} & x_2^{\lambda_2+1} & \dots & x_2^{\lambda_m+m-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_m^{\lambda_1} & x_m^{\lambda_2+1} & \dots & x_m^{\lambda_m+m-1} \end{pmatrix} \quad (4.30)$$

As before the unwanted moves are suppressed automatically.

Again we have the determinant solution which in some sense corresponds to the free fermion model.

Only one thing we have to do is to “quantize” the variables  $x_i$  with the help of the residual equations (4.26) and (4.27).

Substituting the expansion (4.28) to the conditions (4.26) and (4.27) we get the following equations for wave functions:

$$\begin{aligned} \Psi(0, \lambda_2, \dots, \lambda_m) &= e^{2\pi i a/m} \Psi(\lambda_2, \dots, \lambda_m, n), \\ \Psi(\lambda_1, \dots, \lambda_{m-1}, n+1) &= e^{-2\pi i a/m} \Psi(1, \lambda_1, \dots, \lambda_{m-1}). \end{aligned}$$

These equations are formal in the sense that we do not determine what the wave functions  $\Psi(0, \dots)$  and  $\Psi(\dots, n+1)$  are. If we substitute the formula (4.30) into the last pair of the relations we can obtain from the first one:

$$\text{Det} \begin{pmatrix} 1 & x_1^{\lambda_2+1} & \dots & x_1^{\lambda_m+m-1} \\ 1 & x_2^{\lambda_2+1} & \dots & x_2^{\lambda_m+m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m^{\lambda_2+1} & \dots & x_m^{\lambda_m+m-1} \end{pmatrix} = \eta^a \text{Det} \begin{pmatrix} x_1^{\lambda_2} & \dots & x_1^{\lambda_m+m-2} & x_1^{n+m-1} \\ x_2^{\lambda_2} & \dots & x_2^{\lambda_m+m-2} & x_1^{n+m-1} \\ \vdots & \ddots & \vdots & \vdots \\ x_m^{\lambda_2} & \dots & x_m^{\lambda_m+m-2} & x_1^{n+m-1} \end{pmatrix}$$

and from the second one:

$$\text{Det} \begin{pmatrix} x_1^{\lambda_1} & \dots & x_1^{\lambda_{m-1}+m-2} & x_1^{n+m} \\ x_2^{\lambda_1} & \dots & x_2^{\lambda_{m-1}+m-2} & x_1^{n+m} \\ \vdots & \ddots & \vdots & \vdots \\ x_m^{\lambda_1} & \dots & x_m^{\lambda_{m-1}+m-2} & x_1^{n+m} \end{pmatrix} = \eta^{-a} \text{Det} \begin{pmatrix} x_1 & x_1^{\lambda_1+1} & \dots & x_1^{\lambda_{m-1}+m-1} \\ x_2 & x_2^{\lambda_1+1} & \dots & x_2^{\lambda_{m-1}+m-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m^{\lambda_1+1} & \dots & x_m^{\lambda_{m-1}+m-1} \end{pmatrix}$$

For the sake of simplicity we have introduced the following notation:

$$\eta = e^{2\pi i/m}$$

Let us consider the first equality. After some algebra we get:

$$x_i^{n+m} = (-1)^{m+1} \eta^{-a} \prod_{j=1}^m x_j, \quad i = 1, 2, \dots, m. \quad (4.31)$$

It can be easily seen that the second equality gives the same condition. Then we obtain that the relation  $x_k/x_i$  is some root of unity of the power  $m+n$ . Let us introduce the following notation:

$$\omega = e^{\frac{2\pi i}{m+n}} \quad (4.32)$$

We can look for the solution to the equation (4.31) in the following form:

$$x_i = \rho(n, m) \omega^{k_i}, \quad i = 1, 2, \dots, m. \quad (4.33)$$

Here as in the formula (4.14)  $k_i$  are the wave numbers which satisfy the following chain of the inequalities:

$$0 < k_1 < k_2 < \dots < k_m < m+n. \quad (4.34)$$

Substituting it into the formula (4.31) we easily get

$$\rho(n, m) = e^{2\pi i \phi(n, m)}, \quad \phi(n, m) = \frac{1}{n(n+m)} \sum_{i=1}^m k_i - \frac{a}{nm} - \frac{m+1}{2n}. \quad (4.35)$$

We have conjectured that there are no coinciding variables  $x_i$ . We should note that in principle we could also consider the coinciding  $x_i$  by taking the accurate limit. But the calculation of number of states shows us that it is not necessary to do. Indeed, the chain of inequalities (4.34) coincides with (4.17). Hence, after taking into account that  $a = 0, \dots, m-1$  we come to the formula (4.18) for the number of all eigenvectors.

Let us adduce the final result for the eigenvectors of the Hamiltonian  $H$  in case of the string of type (1,1)

$$\Psi(\lambda) = \rho(n, m)^{\lambda_1 + \dots + \lambda_m} \text{Det}(\omega^{k_i(\lambda_j + j - 1)})_{1 \leq i, j \leq m} \quad (4.36)$$

The spectrum of the Hamiltonian is given by

$$E(k_1, \dots, k_m) = \sum_{i=1}^m 2 \cos 2\pi \left( \lambda(n, m) + \frac{k_i}{m+n} \right). \quad (4.37)$$

We can consider the shift operator in horizontal direction  $P$  (or momentum operator) which acts on some quantum state as follows

$$P | \lambda_1, \dots, \lambda_m \rangle = | \lambda_1 + 1, \dots, \lambda_m + 1 \rangle \quad P^n = I. \quad (4.38)$$

Actually, as the operator  $U$  defined by formula (4.23) momentum operator  $P$  commutes with the Hamiltonian. Hence, the eigenvalue of the operator  $P$   $p = \rho(n, m)^m e^{\frac{2\pi i b}{n}}$  where  $b = 0, 1, \dots, n-1$  is the “quantum number”. One can see from the solution (4.36) that

$$b = \sum_{i=1}^m k_i \mod n. \quad (4.39)$$

So, all eigenvalues and eigenvectors belong to the different  $mn$  sectors labeled by two integers  $a = 0, 1, \dots, m-1$  and  $b = 0, 1, \dots, n-1$ .

## 4.5. The string of type (1,2)

We have considered above the spectrum problem for the string with fixed ends and the string of type (1,1) on the torus. The determinant formulae (4.11) and (4.30) for the wave functions have a features of the fermionic free system. In some sense it corresponds to the non-interacting case. Now we are going to consider the case of the string of type (1,2) which is already the example of the interacting string or more exactly self-interacting string. This string winds round the torus one time in horizontal direction and twice in the vertical one. The example of the state of such a string is shown in Fig. 15

We can try to reduce the case of the (1,2)-string to the previous case of the (1,1)-string by applying some simple trick. Namely, let us consider some arbitrary configuration of (1,2)-string on the lattice  $n \times m$ , for example, shown in Fig. 15. Let us also consider the (1,1)-string configuration on the lattice  $n \times 2m$  which consists of two sheets  $n \times m$ .



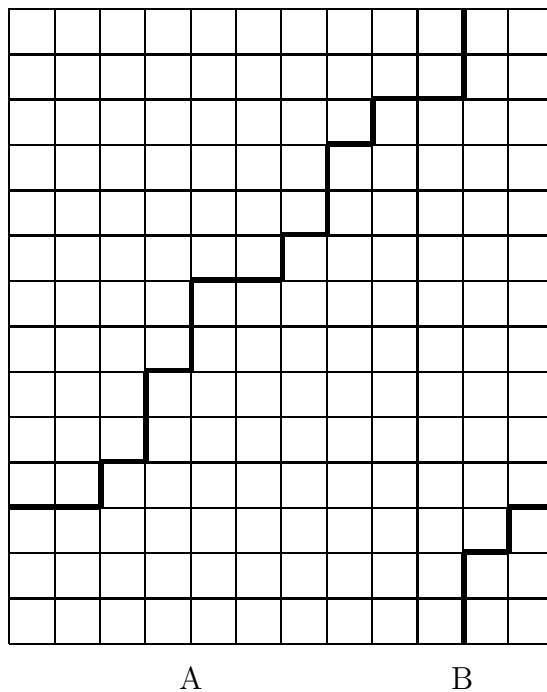


Fig.16 The (1,1)-string corresponding to the initial string of type (1,2)

Nevertheless the quantum problems for the (1,2)-string on the  $n \times m$  lattice and the (1,1)-string on the  $n \times 2m$  lattice are still not completely equivalent. To make them equivalent we should introduce an additional restriction on the (1,1)-string which comes from a simple observation that the length of each “jump” of the (1,1)-string can not be greater than  $m$ . Otherwise we would be forced to lie one part of the string on another when making the back procedure of the comparison of the (1,1)-string to the (1,2)-string. So, we come to the following exclusion rule for the wave function  $\Psi(\lambda)$  of the (1,1)-string

$$\Psi(\lambda) = 0 \quad \text{if} \quad \lambda_i = \lambda_{i+1} = \dots = \lambda_{i+m} \quad \text{for at least one index } i. \quad (4.41)$$

We claim that if this restriction for the (1,1)-string is fulfilled then the quantum problems for the (1,1)-string living on the lattice  $n \times 2m$  and for the (1,2)-string living on the lattice  $n \times m$  are equivalent to each other.

In Appendix we consider the case of the (1,2)-string for  $m = 2$  in more details. Below we shall adduce only the final result for the general case of  $m$ . But first of all let us introduce some useful notation. Let  $D$  be the generalization of the formula (7.12) from the Appendix



$$D(n_1, \dots, n_{2m-1} | k_1, \dots, k_{2m}) = \text{Det} \begin{pmatrix} 1 & \omega^{n_1 k_1} & \omega^{n_2 k_1} & \dots & \omega^{n_{2m-1} k_1} \\ 1 & \omega^{n_1 k_2} & \omega^{n_2 k_2} & \dots & \omega^{n_{2m-1} k_2} \\ \vdots & \vdots & \ddots & & \vdots \\ 1 & \omega^{n_1 k_{2m}} & \omega^{n_2 k_{2m}} & \dots & \omega^{n_{2m-1} k_{2m}} \end{pmatrix} \quad (4.42)$$

where  $\omega = e^{\frac{2\pi i}{n+2m}}$ . Let us also define “partial”  $D$ -functions:

$$D_{i_1, \dots, i_{m-1}}(k_1, \dots, k_{2m}) = D(1, 2, \dots, m, i_1, i_2, \dots, i_{m-1} | k_1, k_2, \dots, k_{2m})$$

$$D^{i_1, \dots, i_{m-1}}(k_1, \dots, k_{2m}) = D_{i_1, \dots, i_{m-1}}(-k_1, \dots, -k_{2m}) \quad (4.43)$$

where  $m+1 < i_1 < \dots < i_{m-1} < n+2m-1$ . Let us also adduce the generalization of the matrix  $A$  given by formula (7.22)

$$A_{i_1, \dots, i_{m-1}}^{j_1, \dots, j_{m-1}}(E) = \sum_{0 \leq k_1 < k_2 < \dots < k_{2m-1} < 2m+n} \frac{D_{i_1, \dots, i_{m-1}}(k_1, \dots, k_{2m}) D^{j_1, \dots, j_{m-1}}(k_1, \dots, k_{2m})}{E - E(k_1, \dots, k_{2m})} \quad (4.44)$$

where  $k_{2m} = b - \sum_{i=1}^{2m-1} k_i$  and  $b = -n+1 + m(2m-3), \dots, m(2m-3)$

$$E(k_1, \dots, k_{2m}) = \rho \sum_{j=1}^{2m} \omega^{k_j} + \rho^{-1} \sum_{j=1}^{2m} \omega^{-k_j},$$

$$\rho = e^{2\pi i \phi}, \quad \phi = \frac{b}{n(n+2m)} - \frac{a}{nm} - \frac{2m-3}{2n},$$

$$a = 0, 1, \dots, m-1. \quad (4.45)$$

Let us note that  $E(k_1, \dots, k_{2m})$  is nothing else but the energy of the Hamiltonian for (1,1)-string<sup>3</sup> Two integers  $a$  and  $b$  connected with the momentums in vertical and horizontal directions respectively are considered to be fixed. It is easy to see that the matrix  $A$  has a dimension  $\binom{n+m-3}{m-1}$ . The spectrum of the Hamiltonian for the case of the (1,2)-string is determined as a solution to the “secular” equation

$$\text{Det} A(E) = 0. \quad (4.46)$$

---

<sup>3</sup>Here we have used another form of the solution to the (1,1)-string problem which is equivalent to that given by formulae (4.33-4.37) up to enumeration of wave numbers  $k_i$  and the parameter  $b$ .

If we have succeeded in finding some it's solution, say  $E^*$ , then there exists a zero vector  $\zeta_{i_1, \dots, i_{m-1}}$  such that  $A\zeta = 0$  and the wave function for the Hamiltonian have the following form:

$$\Psi(0, n_1, \dots, n_{2m-1}) = \rho^{n_1 + \dots + n_{2m-1}} \sum_{m+1 < j_1 < \dots < j_{m-1} < n+2m-1} \zeta_{j_1, \dots, j_{m-1}} \sum_{0 \leq k_1 < k_2 < \dots < k_{2m-1} < 2m+n} \frac{D(n_1, \dots, n_{2m-1} | k_1, \dots, k_{2m}) D^{j_1, \dots, j_{m-1}}(k_1, \dots, k_{2m})}{E - E(k_1, \dots, k_{2m})}, \quad (4.47)$$

where as above  $k_{2m} = b - \sum_{i=1}^{2m-1} k_i$  and  $b = -n+1+m(2m-3), \dots, m(2m-3)$ . In formula (4.47) we use alternative way to determine the wave function which will be described in Appendix. Namely, the wave function depends on the differences  $n_i = \bar{n}_i - \bar{n}_0$  and  $\bar{n}_{i-1} = \lambda_i + i - 1$ . The wave function for other combinations can be easily obtained by shifting the coordinates on some definite number. For example,  $\Psi(r, n_1+r, \dots, n_{2m-1}+r) = p^r \Psi(0, n_1, \dots, n_{2m-1})$  where  $p = \rho^{2m} \omega^b$  is an eigenvalue of the momentum operator  $P$  (see formula (4.38)).

As it was mentioned above the derivation of these formulae for  $m = 2$  will be given in Appendix. But it is not very difficult to understand how they work by the direct checking that the formula (4.47) with the equation (4.46) really gives us the solution to the eigenvalue problem. We have called (4.46) “secular” equation because it has a similar form to the secular equation in quantum mechanics. But in comparison with the ordinary secular equation which provides the second order correction to the energy our “secular” equation being solved provides the rigorous result for the energy of the string of type (1,2). Of course, this is a polynomial equation and we are not able to solve it manifestly. In comparison with the “free” case of the (1,1)-string this case contains some kind of the “diffraction”. Therefore, it seems to be non-integrable.

## 5. Discussion

In this paper we have concentrated on some subspace of the whole quantum space i.e. the space of the “string” states. First of all, we have considered some more simple cases of the string with fixed ends and the string of type

(1,1) living on the torus. Both of these cases appeared to be equivalent to some free fermionic system. We use the wave functions of this free system as a basis functions for the expansion in the case of the string of type (1,2) which is the example of the “self-interacting” string. The energy spectrum is given by the “secular” equation. We hope that the similar result can also be obtained for other types of the single string which has an arbitrary winding numbers round the torus in horizontal and vertical directions. We think that the case of two and more string is not more complicated in comparison with the case of the single string. It could be also reasonable to consider the “scattering” of two and more strings with the fixed ends. Some preliminary analysis shows that the picture of the interaction in case of two strings with the fixed ends is very interesting and looks rather non-trivial.

We also hope that the investigation of the thermodynamic limit can be done for the interacting case as well. Perhaps, it will demand the introduction of some other parameters because the Hamiltonian considered here corresponds to the simplest variant and can be, in principle, generalized by introducing new terms with some arbitrary coefficients as in XXZ or XYZ spin chains.

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## 7. Appendix

Here we consider the case  $m = 2$  of the string of type (1,2). As we claimed above we have to consider the equivalent problem of the (1,1)-string on the lattice  $n \times 4$  with the restriction that not greater than two neighbouring “coordinates” of jumps  $\lambda_i$  can coincide with each other (see formula (4.41)). Let  $|\lambda_1, \lambda_2, \lambda_3, \lambda_4; a\rangle$  where  $a = 0, 1$  and  $1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq n$

is the quantum state obtained by the Fourier transformation as in formula (4.40). It is more convenient to mark these states in terms of the variables  $\bar{n}_{i-1} = \lambda_i + i - 1$  which satisfy

$$0 < \bar{n}_0 < \bar{n}_1 < \bar{n}_2 < \bar{n}_3 < N, \quad (7.1)$$

where  $N = n + 4$ . Namely, dropping again the dependence on  $a$  we get now notation for the quantum state  $|\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3\rangle$ . Let  $P$  be the shift operator in the horizontal direction

$$P |\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3\rangle = |\bar{n}_0 + 1, \bar{n}_1 + 1, \bar{n}_2 + 1, \bar{n}_3 + 1\rangle \quad (7.2)$$

and  $P^n = I$ . Let us also define the state  $|0, n_1, n_2, n_3\rangle$  as follows

$$|0, \bar{n}_1 - \bar{n}_0, \bar{n}_2 - \bar{n}_0, \bar{n}_3 - \bar{n}_0\rangle = P^{-\bar{n}_0} |\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3\rangle. \quad (7.3)$$

So, using this definition we can deal only with states of the form  $|0, n_1, n_2, n_3\rangle$  where  $n_1, n_2, n_3$  are used instead of the differences  $\bar{n}_1 - \bar{n}_0, \bar{n}_2 - \bar{n}_0, \bar{n}_3 - \bar{n}_0$  in the LHS of (7.3). If  $n_1 > 1$  then doing one step down we can easily get

$$|0, n_1, n_2, n_3\rangle = (-1)^a |n_1 - 1, n_2 - 1, n_3 - 1, N - 1\rangle. \quad (7.4)$$

If  $n_1 = 1$  we can do two steps down. Then

$$|0, 1, n_2, n_3\rangle = |n_2 - 2, n_3 - 2, N - 1, N\rangle. \quad (7.5)$$

Shifting consequently the state  $|0, n_1, n_2, n_3\rangle$  on  $n_1 - 1, n_2 - 2$  and  $n_3 - 3$  steps in the horizontal direction and using (7.2) and (7.4, 7.5) we obtain the following chain of the equalities

$$\begin{aligned} |0, n_1, n_2, n_3\rangle &= (-1)^a P^{n_1-1} |0, n_2 - n_1, n_3 - n_1, N - n_1\rangle = \\ &P^{n_2-2} |0, n_3 - n_2, N - n_2, N + n_1 - n_2\rangle = \\ &(-1)^a P^{n_3-3} |0, N - n_3, N + n_1 - n_3, N + n_2 - n_3\rangle. \end{aligned} \quad (7.6)$$

Let us call the states for which any three variables  $n_i, n_{i+1}, n_{i+2}$  go in successive way (for example  $0, 1, 2$ ) “forbidden” states. In fact, we have only four possibilities for the forbidden states. Namely, the states of a types  $|0, 1, 2, n_3\rangle, |0, n_1, n_1 + 1, n_1 + 2\rangle, |0, n_1, N - 2, N - 1\rangle$  and  $|0, 1, n_2, N - 1\rangle$  are forbidden. The wave function must be zero on such a states in accordance with (4.41).

Let  $H_0$  be the Hamiltonian for the (1,1)-string which does not “distinguish” the “forbidden” states from other states satisfying only the requirement (7.1). The Hamiltonian  $H$  for which we want to solve the eigenvalue problem is

$$H = H_0 + \delta H \quad (7.7)$$

where the interaction  $\delta H$  acts non-trivially only on the “forbidden” states. Actually, the action of  $\delta H$  should be so that the result of it's action on some forbidden state would compensate a result of the action of  $H_0$  on this state. We should note that the resulting states can not be forbidden already. We have to take into consideration only such a states. The result of action of  $\delta H$  on the arbitrary state  $|0, n_1, n_2, n_3\rangle$  is a sum of four terms in accordance with four possibilities to get the forbidden states mentioned above. Namely, we have

$$\begin{aligned} \delta H |0, n_1, n_2, n_3\rangle = & -\delta_{n_1,1}\delta_{n_2,2}(P^{-1} |0, 2, 3, n_3 + 1\rangle + |0, 1, 3, n_3\rangle) - \\ & \delta_{n_2,n_1+1}\delta_{n_3,n_1+2}(-1)^a P^{n_1-1}(P^{-1} |0, 2, 3, N - n_1 + 1\rangle + |0, 1, 3, N - n_1\rangle) \\ & - \delta_{n_2,N-2}\delta_{n_3,N-1}(P^{-1} |0, 2, 3, n_1 + 1\rangle + |0, 1, 3, n_1 + 2\rangle) \\ & - \delta_{n_1,1}\delta_{n_3,N-1}(-1)^a(P^{-1} |0, 2, 3, n_2 + 2\rangle + |0, 1, 3, n_2 + 1\rangle) \end{aligned} \quad (7.8)$$

As in Section 4 we are going to solve the eigenvalue problem

$$(H_0 + \delta H) |\Psi\rangle = E |\Psi\rangle \quad (7.9)$$

with the help of the expansion

$$|\Psi\rangle = \sum_{0 < \bar{n}_0 < \bar{n}_1 < \bar{n}_2 < \bar{n}_3 < N} \Psi(\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3) |\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3\rangle, \quad (7.10)$$

where the contribution from the forbidden states are suppressed by the requirement that the wave function  $\Psi(\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3)$  is zero for them.

Now one can apply the technique which is rather standard in quantum mechanics. Namely, we can look for the wave function as an expansion on the basis of the eigenfunctions of the operator  $H_0$  and then use the completeness of these eigenfunctions. So, using the result (4.36) with the different enumeration of the wave numbers  $k_i$  and parameter  $b$  we get

$$\Psi(0, n_1, n_2, n_3) = \sum_{0 \leq k_1 < k_2 < k_3 < N} \rho^{n_1+n_2+n_3} C(k_1, k_2, k_3) D(n_1, n_2, n_3 | k_1, k_2, k_3, k_4), \quad (7.11)$$

where  $k_4 = b - k_1 - k_2 - k_3$  and  $b = -n + 3, \dots, 2$  is supposed to be fixed

$$D(n_1, n_2, n_3 | k_1, k_2, k_3, k_4) = \text{Det} \begin{pmatrix} 1 & \omega^{n_1 k_1} & \omega^{n_2 k_1} & \omega^{n_3 k_1} \\ 1 & \omega^{n_1 k_2} & \omega^{n_2 k_2} & \omega^{n_3 k_2} \\ 1 & \omega^{n_1 k_3} & \omega^{n_2 k_3} & \omega^{n_3 k_3} \\ 1 & \omega^{n_1 k_4} & \omega^{n_2 k_4} & \omega^{n_3 k_4} \end{pmatrix} \quad (7.12)$$

$$\rho = e^{2\pi i \phi}, \quad \phi = \frac{1}{nN} b - \frac{a+1}{2n}, \quad a = 0, 1 \quad (7.13)$$

and  $\omega = e^{\frac{2\pi i}{N}}$ . Substituting (7.11) into the equation (7.9) and taking into account that the eigenvalues of the Hamiltonian  $H_0$  are

$$E(k_1, k_2, k_3, k_4) = \rho \sum_{i=1}^4 \omega^{k_i} + \rho^{-1} \sum_{i=1}^4 \omega^{-k_i} \quad (7.14)$$

we get the equation for the unknown coefficients  $C(k_1, k_2, k_3)$

$$\begin{aligned} & \rho^{n_1+n_2+n_3} \sum_{0 \leq k_1 < k_2 < k_3 < N} \\ & (E - E(k_1, k_2, k_3, k_4)) C(k_1, k_2, k_3) D(n_1, n_2, n_3 | k_1, k_2, k_3, k_4) = \\ & \delta_{n_1,1} \delta_{n_2,2} S(n_3) + \delta_{n_2,n_1+1} \delta_{n_3,n_1+2} (-1)^a p^{n_1-1} S(N - n_1) \\ & + \delta_{n_2,N-2} \delta_{n_3,N-1} S(n_1 + 2) + \delta_{n_1,1} \delta_{n_3,N-1} S(n_2 + 1) \end{aligned} \quad (7.15)$$

where  $S(j) = -p^{-1} \Psi(0, 2, 3, j+1) - \Psi(0, 1, 3, j)$  and  $p = \rho^4 \omega^b$  is eigenvalue of the shift operator  $P$  defined by (7.2). Now we can use the fact that the wave function  $\Psi$  is zero for the forbidden configurations. So, we come to  $S(3) = S(N-1) = 0$ .

Let us multiply the LHS and RHS of the equation (7.15) on

$$\rho^{-n_1-n_2-n_3} D(n_1, n_2, n_3 | -l_1, -l_2, -l_3, -l_4)$$

with  $0 \leq l_1 < l_2 < l_3 < N$  and  $l_4 = b - l_1 - l_2 - l_3$  and take a sum over  $0 \leq n_1 < n_2 < n_3 \leq N - 1$  using the orthohonality property

$$\sum_{0 \leq n_1 < n_2 < n_3 \leq N-1} D(n_1, n_2, n_3 | -l_1, -l_2, -l_3, -l_4) D(n_1, n_2, n_3 | k_1, k_2, k_3, k_4) = \kappa \delta_{k_1, l_1} \delta_{k_2, l_2} \delta_{k_3, l_3} \quad (7.16)$$

where  $\kappa$  is some constant. In this formula we imply that adding to  $k_4$  and  $l_4$  the period  $N$  enough many times  $0 \leq \bar{k}_4 = k_4 + sN \leq N - 1$  and  $0 \leq \bar{l}_4 = l_4 + tN \leq N - 1$  the new numbers  $\bar{k}_4$  and  $\bar{l}_4$  do not coincide with one of  $k_1, k_2, k_3$  and  $l_1, l_2, l_3$  respectively and the sequences  $k_1, k_2, k_3, \bar{k}_4$  and  $l_1, l_2, l_3, \bar{l}_4$  are ordered in the same way, for example,  $k_1 < k_2 < \bar{k}_4 < k_3$  and  $l_1 < l_2 < \bar{l}_4 < l_3$ .

After this we get

$$\begin{aligned} (E - E(l_1, l_2, l_3, l_4)) C(l_1, l_2, l_3) = \\ \sum_{n_3=4}^{N-2} \rho^{-3-n_3} D(1, 2, n_3 | -l_1, -l_2, -l_3, -l_4) S(n_3) \\ \sum_{n_1=2}^{N-4} (-1)^a p^{n_1-1} \rho^{-3-3n_1} D(n_1, n_1+1, n_1+2 | -l_1, -l_2, -l_3, -l_4) S(N-n_1) \\ \sum_{n_1=2}^{N-4} \rho^{-n_1-2N+3} D(n_1, N-2, N-1 | -l_1, -l_2, -l_3, -l_4) S(n_1+2) \\ \sum_{n_2=3}^{N-3} (-1)^a \rho^{-n_2-N} D(1, n_2, N-1 | -l_1, -l_2, -l_3, -l_4) S(n_2+1) \end{aligned} \quad (7.17)$$

After some algebra we come to conclusion that all four terms in the RHS of (7.17) are equal to each other. So, we get the forllowing expression for the coefficients  $C$

$$C(k_1, k_2, k_3) = \sum_{\mu=4}^{N-2} \zeta(\mu) \frac{D(1, 2, \mu | -k_1, -k_2, -k_3, -k_4)}{E - E(k_1, k_2, k_3, k_4)}, \quad (7.18)$$

where  $\zeta(\mu) = 4\rho^{-\mu-3}S(\mu)$ . Substituting this result for  $C$  into the formula (7.11) we get the expression for the wave function

$$\Psi(0, n_1, n_2, n_3) = \rho^{n_1+n_2+n_3} \sum_{0 \leq k_1 < k_2 < k_3 < N} \sum_{\mu=4}^{N-2} \zeta(\mu) \frac{D(n_1, n_2, n_3 | k_1, k_2, k_3, k_4) D(1, 2, \mu | -k_1, -k_2, -k_3, -k_4)}{E - E(k_1, k_2, k_3, k_4)} \quad (7.19)$$

where we imply again that  $k_4 = b - k_1 - k_2 - k_3$ . The last step we should do is to satisfy the requirement that  $\Psi = 0$  for the forbidden configuraions. In fact, due to the cyclicity property it is enough to satisfy only

$$\Psi(0, 1, 2, \mu) = 0. \quad (7.20)$$

So, we come to the condition

$$\sum_{\nu=4}^{N-2} A_{\mu\nu}(E) \zeta(\nu) = 0, \quad (7.21)$$

where  $A$  has the matrix elements

$$A_{\mu\nu}(E) = \sum_{0 \leq k_1 < k_2 < k_3 < N} \frac{D(1, 2, \mu | k_1, k_2, k_3, k_4) D(1, 2, \nu | -k_1, -k_2, -k_3, -k_4)}{E - E(k_1, k_2, k_3, k_4)} \quad \text{for } 4 \leq \mu, \nu \leq N-2$$

where as above we imply that in the sum  $k_4 = b - k_1 - k_2 - k_3$ .

So, the following condition should be valid

$$\text{Det} A(E) = 0. \quad (7.22)$$

This condition can be considered as the equation on the energy  $E$ . The solutions to this equation give us the spectrum for the Hamiltonian  $H$  for the string of type (1,2) in the case  $m = 2$ .

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